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# GENERALIZED CONTINUOUS-TIME RANDOM WALKS, SUBORDINATION BY HITTING TIMES, AND FRACTIONAL DYNAMICS*

V. N. KOLOKOLTSOV†

(*Translated by the author*)

**Abstract.** Functional limit theorems for continuous-time random walks (CTRW) are found in the general case of dependent waiting times and jump sizes that are also position dependent. The limiting anomalous diffusion is described in terms of fractional dynamics. Probabilistic interpretation of generalized fractional evolution is given in terms of the random time change (subordination) by means of hitting times processes.

**Key words.** fractional stable distributions, anomalous diffusion, fractional derivatives, limit theorems, continuous-time random walks, time change, Lévy subordinators, hitting time processes

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**1. Introduction.** Suppose $(X_1, T_1), (X_2, T_2), \ldots$ is a sequence of independent identically distributed pairs of random variables such that $X_i \in \mathbf{R}^d$, $T_i \in \mathbf{R}_+$ (jump sizes and waiting times between the jumps), the distribution of each $(X_i, T_i)$ being given by a probability measure $\psi(dx\,dt)$ on $\mathbf{R}^d \times \mathbf{R}_+$. Let

$$N_t = \max\left\{n\colon\ \sum_{i=1}^{n} T_i \leqq t\right\}.$$

The process

$$S_{N_t} = X_1 + X_2 + \cdots + X_{N_t} \tag{1}$$

is called the continuous-time random walk (CTRW) arising from $\psi$. These CTRWs were introduced in [19] and found numerous applications in physics and economics (see, e.g., [25], [15], [3], [13], [17], and references therein). Of particular interest are the situations where $T_i$ belong to the domain of attraction of a $\beta \in (0,1)$-stable law and $X_i$ belong to the domain of attraction of an $\alpha \in (0,2)$-stable law. The limit distributions of appropriately normalized sums $S_{N_t}$ were first studied in [7] in the case of independent $T_i$ and $X_i$ (see also [11]). In [5] the rate of convergence in double array schemes was analyzed, and in [15] the corresponding functional limit was obtained, which was shown to be specified by fractional differential equations. The basic cases of dependent $T_i$ and $X_i$ were developed in [2] in the framework of the theory of the operator stable processes (see [16] for the latter). Here we extend the theory much further to include possible dependence of $(T_n, X_n)$ on the current position, i.e., to spatially nonhomogeneous situations. Our method is quite different from those used in [7], [11], [16]. It is based on the finite difference approximations to continuous-time operator semigroups and applies the previous results of the author from [8] on stable-like processes.

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†Department of Statistics, University of Warwick, Coventry CV4 7AL, UK, and Moscow Institute of Economics, Moscow, Russia (v.kolokoltsov@warwick.ac.uk).

It was noted in [15] that fractional evolution appears from the subordination of Lévy processes by the hitting times of stable Lévy subordinators. Implicitly this idea was present already in [21]. As a basis for our limit theorems, we develop here the general theory of subordination of Markov processes by the hitting time process showing that this procedure leads naturally to (generalized) fractional evolutions. In particular, in spite of the remark from [15] that the method from [21] (going actually back to [19]) "does not identify the limit process" we shall give a rigorous probabilistic interpretation of the intuitively appealing (but rather formal) calculations from [21].

In the following section we demonstrate our approach to the limits of CTRW by obtaining simple (but nevertheless seemingly new) limit theorems for position-dependent random walks with jump sizes from the domain of attraction of stable laws. In section 3 these results will be extended to double scaled random walks, which are needed for the analysis of CTRW. Section 4 (which is independent of section 2 and may be of independent interest) is devoted to the theory of subordination by hitting times. In section 5 we combine the two bits of the theory from sections 3 and 4, giving our main results on CTRW.

Let us fix some (rather standard) notation to be used throughout the paper. For a locally compact space $X$ we denote by $C(X)$ the Banach space of bounded continuous functions (equipped with the sup-norm) and by $C_\infty(X)$ its closed subspace consisting of functions vanishing at infinity. We denote by $(f,\mu)$ the usual pairing $\int f(x)\mu(dx)$ between functions and measures. By a continuous family of transition probabilities (CFTP) in $X$ we mean as usual a family $p(x;dy)$ of probability measures on $X$ depending continuously on $x \in X$, where probability measures are considered in their weak topology ($\mu_n \to \mu$ as $n \to \infty$ means that $(f,\mu_n) \to (f,\mu)$ as $n \to \infty$ for any $f \in C(X)$).

For a measure $\mu(dy)$ in $\mathbf{R}^d$ and a positive number $h$ we denote by $\mu(dy/h)$ the scaled measure defined via its action

$$\int g(z)\,\mu(dz/h) = \int g(hy)\,\mu(dy)$$

on functions $g \in C(\mathbf{R}^d)$.

The uppercase letters $\mathbf{E}$ and $\mathbf{P}$ are reserved to denote expectation and probability. The function $\delta(x)$ is the usual Dirac function (distribution).

**2. Limit theorems for position-dependent random walks.** For a vector $y \in \mathbf{R}^d$ we shall always denote by $\overline{y}$ its normalization $\overline{y} = y/|y|$, where $|y|$ means the usual Euclidean norm.

Fix an arbitrary $\alpha \in (0,2)$. Let $S\colon\ \mathbf{R}^d \times S^{d-1} \mapsto \mathbf{R}_+$ be a continuous nonnegative function that is symmetric with respect to the second variable, i.e., $S(x,y) = S(x,-y)$. It defines a family of $\alpha$-stable $d$-dimensional symmetric random vectors (depending on $x \in \mathbf{R}^d$) specified by its characteristic function $\phi_x$ with

$$\log \phi_x(p) = \int_0^\infty \int_{S^{d-1}} \left( e^{i(p,\xi)} - 1 - \frac{i(p,\xi)}{1+\xi^2} \right) \frac{d|\xi|}{|\xi|^{1+\alpha}}\, S(x,\overline{\xi})\, d_S\overline{\xi}, \tag{2}$$

where $d_S$ denotes the Lebesgue measure on the sphere $S^{d-1}$. It is well known that it can also be rewritten in the form

$$\log \phi_x(p) = C_\alpha \int_{S^{d-1}} |(p,\overline{\xi})|^\alpha S(x,\overline{\xi})\, d_S\overline{\xi}$$

with a certain constant $C_\alpha$.

*Remark* 1. There are no obstacles for extending our theory to nonsymmetric stable laws. But working with symmetric laws shortens the formulas essentially.

THEOREM 2.1. *Assume*

$$C_1 \leqq \int_{S^{d-1}} |(\overline{p}, s)|^\alpha S(x,s)\, d_S s \leqq C_2$$

*for all* $p$ *with some constants* $C_1$, $C_2$ *and that* $S(x,s)$ *has bounded derivatives with respect to* $x$ *up to and inclusive of order* $q \geqq 3$ (*if* $\alpha < 1$, *the assumption* $q \geqq 2$ *is sufficient*). *Then the pseudodifferential operator*

$$Lf(x) = \log \phi_x\left(\frac{1}{i}\frac{\partial}{\partial x}\right) f(x) = \int_0^\infty \int_{S^{d-1}} (f(x+y) - f(x)) \,\frac{d|y|}{|y|^{1+\alpha}}\, S(x,\overline{y})\, d_S\overline{y} \tag{3}$$

*generates a Feller semigroup* $T_t$ *in* $C_\infty(\mathbf{R}^d)$ *with the space* $C^{q-1}(\mathbf{R}^d) \cap C_\infty(\mathbf{R}^d)$ *being its invariant core.*

This result is proved in [8] and [9].

*Remark* 2. In [8] it is also shown that this semigroup has a continuous transition density (heat kernel), but we do not need it.

Denote by $Z_x(t)$ the Feller process corresponding to the semigroup $T_t$. We are interested here in discrete approximations to $T_t$ and $Z_x(t)$.

We shall start with the following technical result.

PROPOSITION 2.1. *Assume that* $p(x; dy)$ *is a CFTP in* $\mathbf{R}^d$ *from the normal domain of attraction of the stable law specified by* (2). *More precisely assume that for an arbitrary open* $\Omega \subset S^{d-1}$ *with a boundary of Lebesgue measure zero*

$$\int_{|y|>n} \int_{\overline{y} \in \Omega} p(x; dy) \sim \frac{1}{\alpha n^\alpha} \int_\Omega S(x,s)\, d_S s, \qquad n \to \infty, \tag{4}$$

(*i.e., the ratio of the two sides of this formula tends to one as* $n \to \infty$) *uniformly in* $x$. *Assume also that* $p(x, \{0\}) = 0$ *for all* $x$. *Then*

$$\min(1, |y|^2)\, p\left(x; \frac{dy}{h}\right) h^{-\alpha} \longrightarrow \min(1, |y|^2)\, \frac{d|y|}{|y|^{\alpha+1}}\, S(x,\overline{y})\, d_S\overline{y}, \quad h \to 0, \tag{5}$$

*where both sides are finite measures on* $\mathbf{R}^d \setminus \{0\}$ *and the convergence is in the weak sense and is uniform in* $x \in \mathbf{R}^d$. *If* $\alpha < 1$, *then also*

$$\min(1, |y|)\, p\left(x; \frac{dy}{h}\right) h^{-\alpha} \longrightarrow \min(1, |y|)\, \frac{d|y|}{|y|^{\alpha+1}} \int_\Omega S(x,\overline{y})\, d_S\overline{y}, \qquad h \to 0,$$

*holds in the same sense.*

*Remark* 3. As the limiting measure has a density with respect to Lebesgue measure, the uniform weak convergence means simply that the measures of any open set with boundaries of Lebesgue measure zero converge uniformly in $x$.

*Proof.* By (4)

$$\int_{|z|>A} \int_{\overline{z} \in \Omega} p\left(x; \frac{dz}{h}\right) h^{-\alpha} = \int_{|y|>A/h} \int_{\overline{y} \in \Omega} p(x; dy)\, h^{-\alpha} \sim \frac{1}{\alpha A^\alpha} \int_\Omega S(x,s)\, d_S s$$

as $h \to 0$. Hence

$$\int_{A<|z|<B} \int_{\overline{z} \in \Omega} p\left(x; \frac{dz}{h}\right) h^{-\alpha} \longrightarrow \int_A^B \frac{d|z|}{|z|^{\alpha+1}} \int_\Omega S(x,s)\, d_S s.$$

Hence $p(x;dz/h)h^{-\alpha}$ converges weakly to $|z|^{-(\alpha+1)}d|z|S(x,z/|z|)d_S(z/|z|)$ on any set separated from the origin. It is easy to see that (5) follows now from the uniform bound

$$\int_{|y|<\varepsilon} \min(1,|y|^2)\, p\left(x;\frac{dy}{h}\right)h^{-\alpha} \leqq C\varepsilon^{2-\alpha} \tag{6}$$

with a constant $C$. In order to prove (6) let us observe that

$$\int_{|y|>n} p(x;dy) \leqq Cn^{-\alpha},$$

with a constant $C$ uniformly for all $x$ and $n>0$ (in fact it holds for large enough $n$ by (4) and is extended to all $n$, because all $p(x,dy)$ are probability measures). Hence for an arbitrary $\varepsilon<1$ one has

$$\int_{|y|<\varepsilon} \min(1,|y|^2)\, p\left(x;\frac{dy}{h}\right)h^{-\alpha} = \int_{|z|<\varepsilon/h} h^2|z|^2\, p\left(x;\frac{dy}{h}\right)h^{-\alpha}.$$

Representing this integral as the countable sum of the integrals over the regions $\varepsilon/(2^{k+1}h)<y\leqq \varepsilon/(2^k h)$ it can be estimated by

$$\sum_{k=0}^{\infty} h^2\left(\frac{\varepsilon}{2^k h}\right)^2 h^{-\alpha}Ch^{\alpha}\cdot 2^{\alpha(k+1)}\varepsilon^{-\alpha} = \sum_{k=0}^{\infty} C\varepsilon^{2-\alpha}\cdot 2^{\alpha}\cdot 2^{-(2-\alpha)k}.$$

This yields (6), since the sum on the right-hand side converges.

The improvement concerning the case $\alpha<1$ is obtained similarly.

Consider the jump-type Markov process $Z^h(t)$ generated by

$$(L_h f)(x) = \frac{1}{h^{\alpha}}\int (f(x+hy)-f(x))\,p(x;dy). \tag{7}$$

For each $h$ the operator $L_h$ is bounded in $C_\infty(\mathbf{R}^d)$ and hence specifies a Feller semigroup there. The probabilistic interpretation of $Z^h(t)$ is as follows. Starting at a point $x$ one waits a random $h^{-\alpha}$-exponential time $\tau$ (i.e., distributed according to $\mathbf{P}\{\tau>t\}=\exp(-th^{-\alpha})$) and then jumps to $x+hY$, where $Y$ is distributed according to $p(x;dy)$. Then the same repeats starting from $x+hY$, etc. In the case when $p$ does not depend on $x$

$$Z^h(t) = h(Y_1+\cdots+Y_{N_t})$$

is a normalized random walk with the number of jumps $N_t$ being a Poisson process with parameter $h^{-\alpha}$, so that $\mathbf{E}\,N_t = th^{-\alpha}$. In particular, the number of jumps $n=N_t\sim th^{-\alpha}$ for small $h$ so that $Z^h(1)\sim n^{-1/\alpha}(Y_1+\cdots+Y_n)$.

THEOREM 2.2. *The semigroup $T_t^h$ generated by $L_h$ converges to the semigroup $T_t$ generated by $L$. In particular, the corresponding processes converge in the sense of finite-dimensional marginal distributions.*

*Remark* 4. Everywhere in this paper we work with only the convergence of semigroups. However, by the standard results (see, e.g., [6, Theorem 19.25]) for Feller processes this convergence is equivalent to the convergence of the distributions of trajectories in an appropriate Skorokhod space of càdlàg paths.

*Proof.* By (7),

$$(L_h f)(x) = \frac{1}{h^\alpha} \int (f(x+z) - f(x))\, p\left(x; \frac{dz}{h}\right),$$

and by Proposition 2.1 this converges to $Lf(x)$ as $h \to 0$ uniformly in $x$ for $f \in C_\infty(\mathbf{R}^d) \cap C^2(\mathbf{R}^d)$. By a well-known result (see, e.g., [14]) the convergence of the generators on the core of the limiting semigroup implies the convergence of semigroups.

The following result concerns the approximations with a nonrandom number of jumps. Define the process $S_x^h(t) = S_x^h([t])$ (the square brackets denote the integer part of a real number) via

$$S_x^h(0) = x, \quad S_x^h(1) = x + hY_1, \dots, \quad S_x^h(j) = S_x^h(j-1) + hY_j, \dots,$$

where each $Y_j$ is distributed according to $p(S_{j-1}; dy)$. If $p(x; dy)$ does not depend on $x$, then

$$S_x^h(n) = x + h(Y_1 + \cdots + Y_n)$$

is just a standard random walk.

We want to compare the Feller process $Z_x(t)$ on an arbitrary fixed time interval $[0, t_0]$ with the discrete approximations $S_x^h(t/\tau)$, when the number of jumps $n = t/\tau$ is connected with the scaling parameter $h$ by $\tau = h^\alpha$.

THEOREM 2.3. *Under the assumptions of Theorem* 2.1 *and Proposition* 2.1 *for any* $f \in C_\infty(\mathbf{R}^d)$, $\mathbf{E}\, f(S_x^h(t/\tau))$ *converges to* $T_t f(x)$ *uniformly on* $t \in [0, t_0]$, *as* $\tau = h^\alpha \to 0$. *In particular, the processes* $S_x^h(t/\tau)$ *converge to* $Z_x(t)$ *in the sense of finite-dimensional distributions.*

*Proof.* It is enough to prove the required convergence for $f \in C^2(\mathbf{R}^d) \cap C_\infty(\mathbf{R}^d)$ only (by Theorem 2.1). Let such an $f$ be chosen. Denote $f_k(x) = \mathbf{E}\, f(S_x^h(k))$. Then by the Markov property $f_k = R_h^k f$, where the operator $R_h$ is defined via the formula

$$R_h f(x) = \int f(x + hy)\, p(x; dy).$$

Clearly each $R_h$ is a positivity preserving contraction on $C_\infty(\mathbf{R}^d)$. On the other hand, the recurrent equation $f_k = R f_{k-1}$ can be rewritten as

$$\frac{f_k(x) - f_{k-1}(x)}{\tau} = h^{-\alpha} \int (f_{k-1}(x + hy) - f_{k-1}(x))\, p(x; dy). \tag{8}$$

This is a discrete time approximation to the equation

$$\frac{\partial f}{\partial t} = Lf \tag{9}$$

on the functions $f \in C^2(\mathbf{R}^d) \cap C_\infty(\mathbf{R}^d)$ (and differentiable in $t$). Since this scheme is well-posed and stable (as it is solvable uniquely by the contraction $R_h^n$) and the solution to (9) is uniquely defined and preserves the space $C^2(\mathbf{R}^d) \cap C_\infty(\mathbf{R}^d)$ (by Theorem 2.1), it follows by the standard (and easy to prove) general results (see, e.g., [22]) that the solutions to the finite-difference approximation converge to the solution of (9). The theorem is proved.

In the case of $p$ not depending on $x$, Theorem 2.3 turns to the known fact on the convergence of random walks with the distribution of jumps from the domain of normal attraction of a stable law to the corresponding stable Lévy motion.

**3. Double-scaled random walks.** To apply the developed theory to CTRW we shall need a generalization with multiscaled walks that we present now.

We are interested in a process in $\mathbf{R}^d \times \mathbf{R}_+$ specified by the generator

$$\begin{aligned}\mathcal{L}f(x,u) = &\int_0^\infty \int_{S^{d-1}} (f(x+y,u) - f(x,u)) \frac{d|y|}{|y|^{1+\alpha}} S(x,u,\overline{y})\, d_S\overline{y} \\ &+ \int_0^\infty (f(x,u+v) - f(x,u)) \frac{1}{v^{1+\beta}} w(x,u)\, dv.\end{aligned} \tag{10}$$

The following result (and its proof) is a straightforward generalization of Theorem 2.1.

THEOREM 3.1. *Assume*

$$C_1 \leqq \int_{S^{d-1}} |(\overline{p}, s)|^\alpha S(x,u,s)\, d_S s \leqq C_2, \quad C_1 \leqq w(x,u) \leqq C_2$$

*with some constants* $C_1$, $C_2$ *and that* $S(x,s)$ *and* $w(x,u)$ *have bounded derivatives with respect to* $x$ *and* $u$ *up to and inclusive of order* $q \geqq 3$. *Then the pseudodifferential operator* (10) *generates a Feller semigroup* $\mathcal{T}_t$ *in* $C_\infty(\mathbf{R}^d \times \mathbf{R}_+)$ (*continuous functions up to the boundary*) *with the space* $(C^{q-1} \cap C_\infty)(\mathbf{R}^d \times \mathbf{R}_+)$ *being its invariant core and hence a Feller process* $(Y,V)(t)$ *in* $\mathbf{R}^d \times \mathbf{R}_+$.

We shall obtain now the corresponding extension of Theorems 2.2 and 2.3.

THEOREM 3.2. *Assume* $p(x,u;dy\,dv)$ *is a CFTP in* $\mathbf{R}^d \times \mathbf{R}_+$, *which is symmetric with respect to the reflection* $y \mapsto -y$ *and for which*

$$p(x,u;\{0\} \times \mathbf{R}_+) + p(x,u;\mathbf{R}^d \times \{0\}) = 0.$$

*Assume also that the projections belong to the domain of normal attraction of stable laws*; *more precisely, that uniformly in* $(x,u)$

$$\int_{|y|>n} \int_{\overline{y} \in \Omega} p(x,u;dy\,dv) \sim \frac{1}{\alpha n^\alpha} \int_\Omega S(x,u,s)\, d_S s, \qquad n \to \infty, \tag{11}$$

$$\int_{v>n} \int_{|y|>A} p(x,u;dy\,dv) \sim \frac{1}{\beta n^\beta} w(x,u,A), \qquad n \to \infty, \tag{12}$$

*for any* $A \geqq 0$ *with a measurable function* $w$ *of three arguments such that*

$$w(x,u,0) = w(x,u), \quad \lim_{A \to \infty} w(x,u,A) = 0 \tag{13}$$

(*so that* $w(x,u,A)$ *is a measure on* $\mathbf{R}_+$ *for any* $x,u$).

*Consider the jump-type processes generated by*

$$(\mathcal{L}_\tau f)(x,u) = \frac{1}{\tau} \int \Big( f(x + \tau^{1/\alpha} y, u + \tau^{1/\beta} v) - f(x,u) \Big) p(x,u;dy\,dv). \tag{14}$$

*Then the Feller semigroups* $\mathcal{T}_t^h$ *in* $C_\infty(\mathbf{R}^d \times \mathbf{R}_+)$ *of these processes* (*which are Feller, because* $\mathcal{L}_h$ *is bounded in* $C_\infty(\mathbf{R}^d \times \mathbf{R}_+)$ *for any* $h$) *converge to the semigroup* $\mathcal{T}_t$.

*Proof.* As in Proposition 2.1 one deduces from (11), (12) that uniformly in $x$, $u$ as $h \to 0$

$$\min(1,|y|^2) \int_0^\infty p\Big(x,u;\frac{dy\,dv}{h}\Big) h^{-\alpha} \longrightarrow \min(1,|y|^2) \frac{d|y|}{|y|^{\alpha+1}} S(x,\overline{y})\, d_S\overline{y}, \tag{15}$$

$$\min(1,v) \int_{|y|>A} p\Big(x,u;\frac{dy\,dv}{h}\Big) h^{-\beta} \longrightarrow \min(1,v)\, w(x,u,A) \frac{dv}{v^{\beta+1}}. \tag{16}$$

Further, assuming $f \in (C^2 \cap C_\infty)(\mathbf{R}^d \times \mathbf{R}_+)$ and writing

$$\mathcal{L}_\tau f(x,u) = I + II,$$

with

$$\begin{aligned} I &= \frac{1}{\tau}\int (f(x+\tau^{1/\alpha}y,u) - f(x,u))\, p(x,u;dy\,dv) \\ &\quad + \frac{1}{\tau}\int (f(x,u+\tau^{1/\beta}v) - f(x,u))\, p(x,u;dy\,dv), \\ II &= \frac{1}{\tau}\int \Big[(f(x+\tau^{1/\alpha}y, u+\tau^{1/\beta}v) \\ &\qquad - f(x+\tau^{1/\alpha}y,u)) - (f(x,u+\tau^{1/\beta}v) - f(x,u))\Big]\, p(x,u;dy\,dv), \end{aligned}$$

one observes that, as in the proof of Theorem 2.2, (15) and (16) (the latter with $A=0$) imply that $I$ converges to $\mathcal{L}f(x,u)$ uniformly in $x,u$. Thus in order to complete our proof we have to show that the function $II$ converges to zero, as $\tau \to 0$. We have

$$II = \int (g(x+\tau^{1/\alpha}y,u,v) - g(x,u,v))\, p\left(x,u;\frac{dy\,dv}{\tau^{1/\beta}}\right)\frac{1}{\tau},$$

with $g(x,u,v) = f(x,u+v) - f(x,u)$.

By our assumptions on $f$,

$$\begin{aligned} |g(x,u,v)| &\leqq C\min(1,v)\left(\max\left|\frac{\partial f}{\partial u}\right| + \max|f|\right) \leqq \widetilde{C}\min(1,v), \\ \left|\frac{\partial g}{\partial x}(x,u,v)\right| &\leqq C\min(1,v)\left(\max\left|\frac{\partial^2 f}{\partial u\,\partial x}\right| + \max\left|\frac{\partial f}{\partial x}\right|\right) \leqq \widetilde{C}\min(1,v) \end{aligned}$$

with some constants $C$ and $\tilde{C}$. Hence by (16) and (13) for an arbitrary $\varepsilon > 0$ there exists an $A$ such that

$$\int_{|y|>A}\Big(g(x+\tau^{1/\alpha}y,u,v) - g(x,u,v)\Big)\, p\left(x,u;\frac{dy\,dv}{\tau^{1/\beta}}\right)\frac{1}{\tau} < \varepsilon;$$

and on the other hand, for an arbitrary $A$

$$\int_{|y|<A}\Big(g(x+\tau^{1/\alpha}y,u,v) - g(x,u,v)\Big)\, p\left(x,u;\frac{dy\,dv}{\tau^{1/\beta}}\right)\frac{1}{\tau} \leqq \tau^{1/\alpha}A\kappa$$

with a constant $\kappa$ so that $II$ can be made arbitrarily small by first choosing large enough $A$ and then choosing small enough $\tau$.

Define now the process $(Y,V)^\tau_{x,u}(t/\tau) = (Y,V)^\tau_{x,u}([t/\tau])$, where

$$\begin{aligned} &(Y,V)^\tau_{x,u}(0) = (x,u), \quad (Y,V)^\tau_{x,u}(1) = (x+\tau^{1/\alpha}Y_1, u+\tau^{1/\beta}V_1), \dots, \\ &(Y,V)^\tau_{x,u}(j) = (Y,V)^\tau_{x,u}(j-1) + (\tau^{1/\alpha}Y_j, \tau^{1/\beta}V_j), \dots, \end{aligned}$$

each pair $(Y_j,V_j)$ is distributed according to $p((Y,V)^\tau_{x,u}(j-1);dy\,dv)$. If $p(x,u;dy\,dv)$ does not depend on $x,u$, then

$$(Y,V)^\tau_{x,u}(n) = (x,u) + \Big(\tau^{1/\alpha}(Y_1+\cdots+Y_n), \tau^{1/\beta}(V_1+\cdots+V_n)\Big).$$

In view of Theorem 3.2 the following result is obtained by literally the same arguments as Theorem 2.3.

THEOREM 3.3. *Under the assumptions of Theorems* 3.1 *and* 3.2 *the linear contractions* $\mathbf{E}\, f((Y,V)^\tau_{x,u}(t/\tau))$ *in* $C_\infty(\mathbf{R}^d \times \mathbf{R}_+)$ *converge to the semigroup* $\mathcal{T}_t f(x,u)$ *of the process* $(Y,V)(t)$ *uniformly on* $t \in [0,t_0]$, *as* $\tau \to 0$.

**4. Subordination by hitting times and generalized fractional evolutions.** Let $X(u)$, $u \geqq 0$, be a Lévy subordinator, i.e., an increasing i.i.d. càdlàg Feller process (adapted to a filtration on a suitable probability space) with the generator

$$Af(x) = \int_0^\infty (f(x+y) - f(x))\,\nu(dy) + a\,\frac{\partial f}{\partial x}, \tag{17}$$

where $a \geqq 0$ and $\nu$ is a Borel measure on $\{y > 0\}$ such that

$$\int_0^\infty \min(1, y)\,\nu(dy) < \infty.$$

We are interested in the inverse function process or the first hitting time process $Z(t)$ defined as

$$Z_X(t) = Z(t) = \inf\{u\colon\ X(u) > t\} = \sup\{u\colon\ X(u) \leqq t\} \tag{18}$$

which is of course also an increasing càdlàg process. To make our further analysis more transparent (avoiding heavy technicalities of the most general case) we shall assume that there exist $\varepsilon > 0$ and $\beta \in (0,1)$ such that

$$\nu(dy) \geqq y^{1+\beta}\,dy, \qquad 0 < y < \varepsilon. \tag{19}$$

For convenient reference we collect in the next statement (without proofs) the elementary (well-known) properties of $X(u)$.

PROPOSITION 4.1. *Under condition* (19)

(i) *the process $X(u)$ is a.s. increasing at each point*, *i.e.*, *it is not a constant on any finite time interval*;

(ii) *distribution of $X(u)$ for $u > 0$ has a density $G(u, y)$ vanishing for $y < 0$, which is infinitely differentiable in both variables and satisfies the equation*

$$\frac{\partial G}{\partial u} = A^\star G, \tag{20}$$

*where $A^\star$ is the dual operator to $A$ given by*

$$A^\star f(x) = \int_0^\infty (f(x-y) - f(x))\,\nu(dy) - a\,\frac{\partial f}{\partial x};$$

(iii) *if extended by zero to the half-space $\{u < 0\}$*, *the locally integrable function $G(u, y)$ on $\mathbf{R}^2$ specifies a generalized function* (*which is in fact infinitely smooth outside* $(0,0)$) *satisfying* (*in the sense of distribution*) *the equation*

$$\frac{\partial G}{\partial u} = A^\star G + \delta(u)\delta(y). \tag{21}$$

COROLLARY 1. *Under condition* (19)

(i) *the process $Z(t)$ is a.s. continuous and $Z(0) = 0$*;

(ii) *the distribution of $Z(t)$ has a continuously differentiable probability density function $Q(t, u)$ for $u > 0$ given by*

$$Q(t, u) = -\frac{\partial}{\partial u}\int_{-\infty}^t G(u, y)\,dy. \tag{22}$$

*Proof.* (i) follows from Proposition 4.1(i) and for (ii) one observes that

$$\mathbf{P}\{Z(t) \leqq u\} = \mathbf{P}\{X(u) \geqq t\} = \int_t^\infty G(u,y)\,dy = 1 - \int_0^t G(u,y)\,dy,$$

which implies (22) by the differentiability of $G$. Let us stress for clarity that (22) defines $Q(t,u)$ as a smooth function for all $t$ as long as $u > 0$ and $Q(t,u) = 0$ for $t \leqq 0$ and $u > 0$.

THEOREM 4.1. *Under condition* (19) *the density* $Q$ *satisfies the equation*

$$A^\star Q = \frac{\partial Q}{\partial u}, \tag{23}$$

*where* $A^\star$ *acts on the variable* $t$, *and the boundary condition*

$$\lim_{u\to 0_+} Q(t,u) = -A^\star \theta(t), \tag{24}$$

*where* $\theta(t)$ *is the indicator function equal to* 1 (*respectively,* 0) *for positive* (*respectively, negative*) $t$. *If* $Q$ *is extended by* 0 *to the half-space* $\{u < 0\}$, *it satisfies the equation*

$$A^\star Q = \frac{\partial Q}{\partial u} + \delta(u) A^\star \theta(t) \tag{25}$$

*in the sense of distribution* (*generalized functions*).

*Moreover the* (*pointwise*) *derivative* $\frac{\partial Q}{\partial t}$ *also satisfies* (23) *for* $u > 0$ *and satisfies the equation*

$$A^\star \frac{\partial Q}{\partial t} = \frac{\partial}{\partial u}\frac{\partial Q}{\partial t} + \delta(u)\frac{d}{dt}A^\star\theta(t) \tag{26}$$

*in the sense of distributions.*

*Remark* 5. In the case of a $\beta$-stable subordinator $X(u)$ with the generator

$$Af(x) = -\frac{1}{\Gamma(-\beta)}\int_0^\infty (f(x+y) - f(x))\,y^{-1-\beta}\,dy \tag{27}$$

one has

$$A = -\frac{d^\beta}{d(-x)^\beta}, \quad A^\star = -\frac{d^\beta}{dx^\beta} \tag{28}$$

(these equations can be considered as the definitions of fractional derivatives; we refer to [18] and [20] for a general background in fractional calculus; a short handy account is also given in the appendix to [21]), in which case (25) takes the form

$$\frac{d^\beta Q}{dt^\beta} + \frac{\partial Q}{\partial u} = \delta(u)\frac{t^{-\beta}}{\Gamma(1-\beta)}, \tag{29}$$

coinciding with (B14) from [21]. This remark gives rise to the idea of calling the operator (17) a generalized fractional derivative.

*Proof.* Notice that by (22) and (20), and by the commutativity of the integration and $A^\star$, one has

$$\begin{aligned} Q(t,u) &= -\int_{-\infty}^t \frac{\partial}{\partial u} G(u,y)\,dy = -\int_{-\infty}^t (A^\star G(u,\cdot))(y)\,dy \\ &= -A^\star \int_{-\infty}^t G(u,y)\,dy. \end{aligned}$$

This implies (23) (by differentiating with respect to $u$ and again using (22)) and (24), because $G(0, y) = \delta(y)$.

Assume now that $Q$ is extended by zero to $\{u < 0\}$. Let $\phi$ be a test function (infinitely differentiable with a compact support) in $\mathbf{R}^2$. Then in the sense of distribution

$$\left(\left(\frac{\partial}{\partial u} - A^\star\right)Q, \phi\right) = \left(Q, \left(-\frac{\partial}{\partial u} - A\right)\phi\right)$$
$$= \lim_{\varepsilon\to 0}\int_\varepsilon^\infty du \int_{\mathbf{R}} dt\, Q(t,u)\left(-\frac{\partial}{\partial u} - A\right)\phi(t,u)$$
$$= \lim_{\varepsilon\to 0}\left[\int_\varepsilon^\infty du \int_{\mathbf{R}} dt\, \phi(t,u)\left(\frac{\partial}{\partial u} - A^\star\right)Q(t,u) + \int_{\mathbf{R}} \phi(t,\varepsilon)\, Q(t,\varepsilon)\, dt\right].$$

The first term here vanishes by (23). Hence by (24)

$$\left(\left(\frac{\partial}{\partial u} - A^\star\right)Q, \phi\right) = -\int_{\mathbf{R}} \phi(t,0)\, A^\star\theta(t)\, dt,$$

which clearly implies (25). The required properties of $\frac{\partial Q}{\partial t}$ follow similarly from the representation

$$\frac{\partial Q}{\partial t}(t,u) = -\frac{\partial G}{\partial u}(u,t).$$

For instance for $u > 0$

$$A^\star \frac{\partial Q}{\partial t}(t,u) = -\frac{\partial}{\partial u} A^\star G(u,t) = -\frac{\partial}{\partial u}\frac{\partial}{\partial u} G(u,t) = \frac{\partial}{\partial u}\frac{\partial Q}{\partial t}.$$

*Remark* 6. Let us stress that the generalized function $Q$ coincides with an infinitely differentiable function outside the ray $\{t \geqq 0,\ u = 0\}$, vanishes on the set $\{t < 0,\ u < 0\}$, and satisfies the limiting condition $\lim_{t\to 0_+} Q(t,u) = \delta(u)$. The latter holds, since for $t > 0$ and a smooth test function $\phi$

$$\int_{-\infty}^\infty Q(t,u)\,\phi(u)\, du = \int_0^\infty du \frac{\partial}{\partial u}\int_t^\infty G(u,y)\, dy\, \phi(u)$$
$$= -\int_0^\infty du\, \phi'(u) \int_t^\infty G(u,y)\, dy,$$

and this tends to $-\int_0^\infty \phi'(u)\, du = \phi(0)$ as $t \to 0$.

We are interested now in the random time change of Markov processes specified by the process $Z(t)$.

THEOREM 4.2. *Under the conditions of Theorem* 4.1 *let* $Y(t)$ *be a Feller process in* $\mathbf{R}^d$, *independent of* $Z(t)$, *and with the domain of the generator* $L$ *containing* $(C_\infty \cap C^2)(\mathbf{R}^d)$. *Denote the transition probabilities of* $Y(t)$ *by*

$$T(t,x,dy) = \mathbf{P}\{Y_x(t) \in dy\} = \mathbf{P}_x\{Y(t) \in dy\}.$$

*Then the distributions of the* (*time changed or subordinated*) *process* $Y(Z(t))$ *for* $t > 0$ *are given by*

$$\mathbf{P}_x\{Y(Z(t)) \in dy\} = \int_0^\infty T(u,x,dy)\, Q(t,u)\, du, \tag{30}$$

*the averages* $f(t,x) = \mathbf{E}\, f(Y_x(Z(t)))$ *of* $f \in (C_\infty \cap C^2)(\mathbf{R}^d)$ *satisfy the* (*generalized*) *fractional evolution equation*

$$A_t^\star f(t,x) = -L_x f(t,x) + f(x)\, A^\star \theta(t) \tag{31}$$

(*where the subscripts indicate the variables on which the operators act*), *and their time derivatives* $h = \partial f/\partial t$ *satisfy for* $t > 0$ *the equation*

$$A_t^\star h = -L_x h + f(x)\, \frac{d}{dt}\, A^\star \theta(t). \tag{32}$$

*Moreover, if* $Y(t)$ *has a smooth transition probability density so that* $T(t,x,dy) = T(t,x,y)\, dy$ *and the forward and backward equations*

$$\frac{\partial T}{\partial t}\,(t,x,y) = L_x T(t,x,y) = L_y^\star T(t,x,y) \tag{33}$$

*hold, then the distributions of* $Y(Z(t))$ *have smooth density*

$$g(t,x,y) = \int_0^\infty T(u,x,y)\, Q(t,u)\, du, \tag{34}$$

*satisfying the forward* (*generalized*) *fractional evolution equation*

$$A_t^\star g = -L_y^\star g + \delta(x-y)\, A^\star \theta(t) \tag{35}$$

*and the backward* (*generalized*) *fractional evolution equation*

$$A_t^\star g = -L_x g + \delta(x-y)\, A^\star \theta(t) \tag{36}$$

(*when* $g$ *is continued by zero to the domain* $\{t < 0\}$) *with the time derivative* $h = \partial g/\partial t$ *satisfying the equation*

$$A_t^\star h = -L_y^\star h + \delta(x-y)\, \frac{d}{dt}\, A^\star \theta(t). \tag{37}$$

*Remark* 7. In the case of a $\beta$-stable Lévy subordinator $X(u)$ with the generator (27), where (28) holds, the left-hand sides of the above equations become fractional derivatives by themselves. In particular, if $Y(t)$ is a symmetric $\alpha$-stable Lévy motion, (35) takes the form

$$\frac{\partial^\beta}{\partial t^\beta}\, g(t, y-x) = \frac{\partial^\alpha}{\partial |y|^\alpha}\, g(t,y-x) + \delta(y-x)\, \frac{t^{-\beta}}{\Gamma(1-\beta)}, \tag{38}$$

deduced in [21] and [23]. The corresponding particular case of (34) also appears in [15] as well as in [21], where it is called a formula of separation of variables. Our general approach makes it clear that this separation of variables comes from the independence of $Y(t)$ and the subordinator $X(u)$ (see Theorem 4.3 for a more general situation).

*Proof.* For a continuous bounded function $f$ one has for $t > 0$ that

$$\begin{aligned} \mathbf{E}\, f(Y_x(Z(t))) &= \int_0^\infty \mathbf{E}\Big( f(Y_x(Z(t)))\,|\, Z(t) = u\Big) Q(t,u)\, du \\ &= \int_0^\infty \mathbf{E}\, f(Y_x(u))\, Q(t,u)\, du \end{aligned}$$

by the independence of $Z$ and $Y$. This implies (30) and (34).

From Theorem 4.1 it follows that for $t > 0$

$$A_t^\star g = \lim_{\varepsilon \to 0} \int_\varepsilon^\infty T(u,x,y)\, A_t^\star Q(t,u)\, du = \lim_{\varepsilon \to 0} \int_\varepsilon^\infty T(u,x,y)\, \frac{\partial}{\partial u}\, Q(t,u)\, du$$
$$= -\int_0^\infty \frac{\partial}{\partial u}\, T(u,x,y)\, Q(t,u)\, du + \delta(x-y)\, A^\star \theta(t),$$

where by (33) the first term equals $-L_y^\star g = L_x g$, implying (35) and (36). Of course for $t < 0$ both sides of (35) and (36) vanish. Other equations are proved analogously. Theorem 4.1 is proved.

Now we want to generalize this theory to the case of Lévy-type subordinators $X(u)$ specified by the generators of the form

$$Af(x) = \int_0^\infty (f(x+y) - f(x))\, \nu(x, dy) + a(x)\, \frac{\partial f}{\partial x} \tag{39}$$

with position depending on Lévy measure and drift. We need some regularity assumptions in order to have a smooth transition probability density like in the case of the Lévy motions.

PROPOSITION 4.2. *Assume that*

(i) *$\nu$ has a density $\nu(x,y)$ with respect to Lebesgue measure such that*

$$C_1 \min\left(y^{-1-\beta_1}, y^{-1-\beta_2}\right) \leqq \nu(x,y) \leqq C_2 \max\left(y^{-1-\beta_1}, y^{-1-\beta_2}\right) \tag{40}$$

*with some constants $C_1, C_2 > 0$, and $0 < \beta_1 < \beta_2 < 1$;*

(ii) *$\nu$ is thrice continuously differentiable with respect to $x$ with the derivatives satisfying the right estimate* (40);

(iii) *$a(x)$ is nonnegative with bounded derivatives up to the order three.*

*Then the generator* (39) *specifies an increasing Feller process having for $u > 0$ a transition probability density $G(u,y) = \mathbf{P}\{X(u) \in dy\}$ (we assume that $X(u)$ starts at the origin) that is twice continuously differentiable in $u$.*

*Remark* 8. Condition (40) holds for popular stable-like processes with a position-dependent stability index.

*Proof.* The existence of the Feller process is proved under much more general assumptions in [1]. A proof of the existence of a smooth transition probability density is given in [8] under slightly different assumptions (symmetric multidimensional stable-like processes), but is easily seen to be valid in the present situation.

One can see now that the hitting time process defined by (18) with $X(u)$ from the previous proposition is again continuous and has a continuously differentiable density $Q(t,u)$ for $t > 0$ given by (22). However, (23) does not hold, since the operators $A$ and integration do not commute. On the other hand, (26) remains true (as is easily seen from the proof). This leads directly to the following partial generalization of Theorem 4.2.

PROPOSITION 4.3. *Let $Y(t)$ be the same Feller process in $\mathbf{R}^d$ as in Theorem* 4.3, *but let independent hitting time process $Z(t)$ be constructed from $X(u)$ under the assumptions of Proposition* 4.2.

*Then the distributions of the (time changed or subordinated) process $Y(Z(t))$ for $t > 0$ are given by* (30), *and the time derivatives $h = \partial f/\partial t$ of the averages $f(t,x) = \mathbf{E}\, f(Y_x(Z(t)))$ of continuous bounded functions $f$ satisfy* (37).

Finally we want to extend this to the case of dependent hitting times.

THEOREM 4.3. *Let* $(Y, V)(t)$ *be a random process in* $\mathbf{R}^d \times \mathbf{R}_+$ *such that*

(i) *the components* $Y(t)$, $V(s)$ *at different times have a joint probability density*

$$\phi(s, u; y_0, v_0; y, v) = \mathbf{P}_{(y_0,v_0)}\{Y(s) \in dy, V(u) \in dv\} \tag{41}$$

*that is continuously differentiable in* $u$ *for* $u, s > 0$;

(ii) *the component* $V(t)$ *is increasing and is a.s. not a constant on any finite interval. For instance, the process from Theorem* 3.1 *enjoys these properties.*

*Then*

(i) *the hitting time process* $Z(t) = Z_V(t)$ (*defined by* (18) *with* $V$ *instead of* $X$) *is a.s. continuous*;

(ii) *there exists a continuous joint probability density of* $Y(s)$, $Z(t)$ *given by*

$$g_{Y(s),Z(t)}(y_0, 0; y, u) = \frac{\partial}{\partial u}\int_t^\infty \phi(s, u; y, v)\, dv; \ \ \textit{and} \tag{42}$$

(iii) *the distribution of the composition* $Y(Z(t))$ *has the probability density*

$$\begin{aligned}\Phi_{Y(Z(t))}(y) &= \int_0^\infty g_{Y(s),Z(t)}(y_0, 0; y, s)\, ds \\ &= \int_0^\infty \left(\frac{\partial}{\partial u}\int_t^\infty \phi(s, u; y_0, 0; y, v)\, dv\right)\Bigg|_{u=s} ds;\end{aligned} \tag{43}$$

(iv) *in particular, if* $(Y, V)(t)$ *is a Feller process with a transition probability density* $G_{YV}(u, y_0, v_0, y, v)$ *and a generator of the form* $(L + A)f(y, v)$, *where* $L$ *acts on the variable* $y$ *and does not depend on* $v$ (*intuition*; *jumps do not depend on waiting times*) *and* $A$ *acts on* $v$ (*but may depend on* $y$), *then for* $s \geqq u$

$$\phi(s, u; y_0, v_0; y, v) = \int G_Y(s - u, z, y)\, G_{YV}(u, y_0, v_0; z, v)\, dz, \tag{44}$$

*where* $G_Y$ *denotes of course the transition probability density of the component* $Y$, *and*

$$\frac{\partial}{\partial t}\Phi_{Y(Z(t))}(y) = \int_0^\infty A^\star G_{YV}(u, y_0, 0; y, t)\, du; \tag{45}$$

*i.e.*, *the time derivative of the density of the subordinated process equals the generalized fractional derivative of the "time component* $V$*" of the integrated joint density of the process* $(Y, V)$. *This derivative* $h = \frac{\partial}{\partial t}\Phi$ *satisfies, instead of* (37), *the more complicated equation*

$$(A^\star + L^\star)\, h = \delta\,(y - y_0)\, A^\star\delta(v) + [L^\star, A^\star]\int_0^\infty G_{YV}(u, y_0, 0; y, v)\, du. \tag{46}$$

*Proof.* Statements (i) and (ii) are straightforward extensions of Corollary 1 to Proposition 4.1. Statement (iii) follows from conditioning and the definition of the joint distribution. To prove (iv) we write for $s \geqq u$ by conditioning to time $u$

$$\begin{aligned}\mathbf{E}\, f(Y_{y_0}(s), V_{(y_0,v_0)}(u)) &= \mathbf{E}\int G_Y(s - u, Y_{y_0}(u); y)\, f(y, V_{(y_0,v_0)})\, dy \\ &= \int\int G_Y(s - u, z; y)\, f(y, v)\, G_{YV}(u, y_0, v_0; z, v)\, dy\, dz\, dv,\end{aligned}$$

implying (44). Consequently,

$$\begin{aligned}\frac{\partial}{\partial t}\,\Phi_{Y(Z(t))}(y) &= -\int_0^\infty \frac{\partial}{\partial u}\int G_Y(s-u,z,y)\,G_{YV}(u,y_0,0;z,t)\,dz\Big|_{u=s}\,ds\\ &= \int_0^\infty \int -(L_z G_Y(s-u,z,y))\,G_{YV}(u,y_0,0;z,t)\,dz\Big|_{u=s}\,ds\\ &\quad+\int_0^\infty \int G_Y(s-u,z,y)\,(A^\star+L^\star)\,G_{YV}(u,y_0,0;z,t)\,dz\Big|_{u=s}\,ds,\end{aligned}$$

which yields (45), since $L$ cancels due to the assumptions on the form of the generator. Finally (45) implies (46) by straightforward inspection.

## 5. Limit theorems for position-dependent CTRW.

Now everything is ready for our main result.

THEOREM 5.1. *Under the assumptions of Theorems* 3.1 *and* 3.2, *let* $Z^\tau(t), Z(t)$ *be the hitting time processes for* $V^\tau(t/\tau)$ *and* $V(t)$, *respectively* (*defined by the corresponding formula* (18)). *Then the subordinated processes* $Y^\tau(Z^\tau(t)/\tau)$ *converge to the subordinated process* $Y(Z(t))$ *in the sense of marginal distributions*, *i.e.*,

$$\mathbf{E}_{x,0}\, f\left(Y^\tau\left(\frac{Z^\tau(t)}{\tau}\right)\right) \longrightarrow \mathbf{E}_{x,0}\, f\big(Y(Z(t))\big), \qquad \tau\to 0, \tag{47}$$

*for arbitrary* $x\in\mathbf{R}^d$, $f\in C_\infty(\mathbf{R}^d\times\mathbf{R}_+)$, *uniformly for* $t$ *from any compact interval.*

*Remark* 9. 1. The distribution of the limiting process is described in Theorem 4.3.

2. We show the convergence in the weakest possible sense. It does not seem difficult to extend it to the convergence in the Skorokhod space of trajectories using standard tools (compactness, etc.) or the theory of continuous compositions from [24].

3. A similar result holds for the continuous time approximation from Theorem 3.2.

*Proof.* Since the time is effectively discrete in $V^\tau(t/\tau)$, it follows that $Z^\tau(t) = \max\{u\colon X(u) \leqq t\}$ and that the events $(Z^\tau(t)\leqq u)$ and $(V^\tau(u/\tau)\geqq t)$ coincide, which implies that the convergence of the finite-dimensional distributions of $(Y^\tau(s/\tau), V^\tau(u/\tau))$ to $(Y(s), V(u))$ (proved in Theorem 3.3) is equivalent to the corresponding convergence of the distributions of $(Y^\tau(s/\tau), Z^\tau(t))$ to $(Y(s), Z(t))$.

Further, since $V(0)=0$, $V(t)$ is continuous and $V(u)\to\infty$ as $u\to\infty$, and because the limiting distribution is absolutely continuous, to show (47) it is sufficient to show that

$$\mathbf{P}_{x,0}\left\{Y^\tau\left(\frac{Z_K^\tau(t)}{\tau}\right)\in A\right\} \longrightarrow \mathbf{P}_{x,0}\{Y(Z_K(t))\in A\}, \qquad \tau\to 0, \tag{48}$$

for large enough $K>0$ and any compact set $A$, whose boundary has Lebesgue measure zero, where

$$Z_K^\tau(t) = Z^\tau(t), \quad K^{-1}\leqq Z^\tau(t)\leqq K,$$

and vanishes otherwise, and similarly $Z_K(t)$ is defined.

Now

$$\mathbf{P}\Big\{Y^\tau\Big(\frac{Z_K^\tau(t)}{\tau}\Big) \in A\Big\}$$

$$= \sum_{k=1/(K\tau)}^{K/\tau} \mathbf{P}\Big\{V^\tau(k) \in A \,\&\, Z^\tau(t) \in (k\tau, (k+1)\tau)\Big\}, \tag{49}$$

$$\mathbf{P}\{Y(Z_K(t)) \in A\} = \sum_{k=1/(K\tau)}^{K/\tau} \int_A dy \int_{k\tau}^{(k+1)\tau} g_{Y(s),Z(t)}(y,s)\,ds. \tag{50}$$

The right-hand side of (50) can be rewritten as

$$\sum_{k=1/(K\tau)}^{K/\tau} \int_A dy \int_{k\tau}^{(k+1)\tau} g_{Y(\tau k),Z(t)}(y,s)\,ds$$

$$+ \sum_{k=1/(K\tau)}^{K/\tau} \int_A dy \int_{k\tau}^{(k+1)\tau} (g_{Y(s),Z(t)} - g_{Y(\tau k),Z(t)})(y,s)\,ds. \tag{51}$$

The second term here tends to zero as $\tau \to 0$ due to the continuity of the function (42), and the difference between the first term and (49) tends to zero, because the distributions of $(Y^\tau(s/\tau), Z^\tau(t))$ converge to the distribution of $(Y(s), Z(t))$. Hence (48) follows. The theorem is proved.

In the case when $S$ does not depend on $u$ and $w$ does not depend on $x$ in (10), the limiting process $(Y, V)(t)$ has independent components so that the averages of the limiting subordinated process satisfy the generalized fractional evolution equation from Proposition 4.3, and if, moreover, $w$ is a constant, they satisfy the fractional equations from Theorem 4.2. In particular, if $p(x, u, dy\,dv)$ does not depend on $(x, u)$ and decomposes into a product $p(dy)q(dv)$, and the limit $V(t)$ is stable, we recover the main result from [15] (in a slightly less general setting, since we worked with symmetric stable laws and not with operator stable motions as in [15]), as well as of course the corresponding results from [7], [11] (put $t = 1$ in (47)) on the long time behavior of the normalized subordinated sums (1).

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